DESIGN OF REINFORCEMENT IN CONCRETE SHELLS: A UNIFIED APPROACH

Stefan J. Medwadowski[a] and Avelino Samartín[b]

[a]Ex-President of the IASS. 5343 Broadway Terrace, no. 205. Oakland, Ca 94618. USA
[b]Professor, ETSI Caminos, Canales y Puertos. Technical University of Madrid. SPAIN

Editor's Note: This paper is a first version of a chapter for a State-of-the-Art Report by IASS Working Group 5: Concrete Shell Roofs. The paper is open for discussion and comment, which may be sent to the Editor of the Journal and Chair of the Working Group, J. F. Abel, jfa5@cornell.edu.

SUMMARY

The problem of design/verification of reinforcement in concrete shells is reviewed. Methods of analysis are classified, and the elastic-plastic *approach is described in detail in the general case of shells subjected to both bending and membrane action. The procedure is then reduced to membrane shells (applicable also to concrete walls) and to pure bending, as in the case of plates. The procedure, which is based on previous research, generally requires the use of a desk-top computer.*

Keywords: *Concrete Shells, design of reinforcement*

1. INTRODUCTION

The subject of this overview is that stage of the design process of concrete thin shells during which the layout, size and amount of reinforcement is determined.

The objective is to ensure an adequate factor of safety associated with the reinforcement and related to any possible limit state appropriate to a given shell. The most common of these limit states is, of course, that of strength.

This overview is organized as follows.

- Stages of shell design
- Layout and size of reinforcement
- Design of reinforcement
  - Historical notes
  - General case: Bending-membrane reinforcement
  - Membrane state of stress only
  - Bending state of stress only
- Concluding remarks
- References

2. STAGES OF SHELL DESIGN

The following stages in the design of concrete shell roofs can be identified.

1. Determination of the shell form, its supports and loads (limit states)
2. Analysis of internal stress resultants and displacements
3. Design/verification of shell reinforcement
4. Verification of the adequacy of concrete material and thickness

The values of internal stress resultants obtained in the step 2 are necessary to perform the design of reinforcement of step 3.

Traditionally, the analysis of a concrete shell is based on the assumption that the shell material is linearly elastic, isotropic and homogeneous. The intensity of the load system is assumed at the level corresponding to working loads. The type of analysis is said to be *elastic*. This approach is the most common in current practice.

It is also possible, at the outset of the analysis, to recognize that reinforced concrete is, in fact, a non-linear, non-homogeneous and anisotropic material, with concrete itself cracking when subjected to tension. The response of the shell is, then, non-linear. Currently, this approach to analysis is used relatively rarely, when it is necessary to study the response of a shell in the non-linear range. The analysis based on these assumptions is termed *plastic* or *material nonlinear analysis*.

Another possible approach, which will not be discussed here, is the *geometrically nonlinear analysis*, in which the material can be linear or nonlinear. Shell buckling is an example of the application of this important analysis.

Similarly, in the design of reinforcement, one can assume that concrete and steel will behave linearly, although concrete will crack if subjected to tension. Such solution is termed *elastic*. If concrete and steel are assumed to be inelastic, the resulting solution is termed *plastic*. Thus, it is possible to distinguish the following three meaningful combinations of analysis and reinforcement design.

- *Elastic-elastic*; this approach is no longer used.
- *Elastic-plastic*; this is the approach widely used in all reinforced concrete design; it corresponds to the strength (ultimate load) design.
- *Plastic-plastic*; used for nonlinear problems, which require special attention.

3. LAYOUT AND SIZE OF REINFORCEMENT

Prior to definitive verification of the reinforcement design, it is necessary to postulate a trial arrangement of the reinforcing. Simplified methods based on beam or beam-column design may be used to help in this process. The following items need be considered.

- The number of curtains of reinforcement: It is generally advisable to place steel in two curtains. one near each surface of the shell. The advantage of this arrangement is greater resistance to flexure, and a reduction in the number and size of surface cracks.
- The number of bar directions in a curtain: Minimum two, sometimes more, depending on the magnitude of internal stress resultants. Ideally, bar directions should coincide with the directions of the principal stresses. This is usually not practical, and bar directions generally depend on the geometry of the shell.

  As an example, bars in cylindrical shell covering a rectangular planform might run in directions parallel to the generatrix (curved bars) and directrix (straight bars).
- The size of bars: Generally smaller size closer spaced bars are preferred.

4. DESIGN OF REINFORCEMENT

The approach to design adopted here is elastic-plastic, i.e., the shell stress resultants are computed from an elastic analysis of the structure, while the design of the reinforcing takes into account the inelastic behavior of steel and concrete. Given an initial postulated layout and size of the reinforcing and the results of an elastic analysis under various design load cases, the designer can use the method presented to evaluate the performance and adequacy of the reinforcing at a selected point on the surface of the shell. The designer needs to select a number of key points at which to carry out this evaluation.

4.1. Historical notes

As noted, the basic assumptions of the *plastic* formulation of the problem of design of reinforcement are (1) concrete and steel are both nonlinear materials, (2) tension is resisted only by reinforcement, since concrete cracks when subjected to tension, and (3) there is no transfer of shears along the plane of the crack. Then, the current procedure for the design of reinforcement can be categorized as *plastic*.

The design of reinforcement in linear elements (columns, beams, etc.) was developed very early. This was not the case for two-dimensional elements: plates, walls, and shells. The first satisfactory solution was obtained by Nielsen [9] for the relatively simple case of in-plane state of stress in a flat plate. Nielsen's results are in qualitative agreement with the experimental results of Vecchio and Collins [10]. They were further developed and extended to membrane reinforcement in shells by Gupta [3], [4] and Medwadowski [7], [8]. Thus, the plastic design of membrane shell reinforcement has been available to the profession for several decades.

The equivalent problem of a general shell, including bending, resisted all efforts to obtain a tractable solution. However, very recently, García and Samartin [1], and Samartín [2], succeeded by using a layered model of a shell element.

In the following, the general case of a shell subjected to both bending and membrane stresses is discussed first. Next, the general equations are specialized to the important in practical applications case of a shell in a membrane state of stress. Finally, the general equations are specialized to the case of pure bending. The last condition is not important in the case of shells, since a well designed shell should never be in the state of pure bending. However, it is very useful in the case of plates, to which it is directly applicable.

4.2. Assumptions

The following assumptions are restated.

- Concrete and steel are both nonlinear materials
- Concrete cracks when subjected to tension, which is resisted only by the reinforcement
- No shears are transferred along the surface of the crack
- Secondary effects, such as strain hardening, or the dowel action of the bars, are considered small and are disregarded for the sake of clarity.

4.3. Constitutive equations

The material constitutive equations, i.e., the stress-strain relations, are assumed for each material, concrete and steel, to be as follows [5].

Concrete

Global equations (1)

$$\sigma_c = \sigma_c(\varepsilon_c) = \begin{cases} 0 & \varepsilon_c < 0 \\ f_c^* \dfrac{\varepsilon_c}{\varepsilon_{c0}}\left(2 - \dfrac{\varepsilon_c}{\varepsilon_{c0}}\right) & 0 \le \varepsilon_c \le \varepsilon_{c0} \\ f_c^* \left[1 - (1-\beta)\dfrac{\varepsilon_c - \varepsilon_{c0}}{\varepsilon_{cu} - \varepsilon_{c0}}\right] & \varepsilon_{c0} \le \varepsilon_c \le \varepsilon_{cu} \\ 0 & \varepsilon_{cu} < \varepsilon_c \end{cases}$$

Incremental tangential equations (2)

$$\frac{d\sigma_c}{d\varepsilon_c} = E_{tc}(\varepsilon_c) = \begin{cases} 0 & \varepsilon_c < 0 \\ E_{ci}\left(1 - \dfrac{\varepsilon_c}{\varepsilon_{c0}}\right) & 0 \le \varepsilon_c \le \varepsilon_{c0} \\ -\dfrac{1-\beta}{\varepsilon_{cu} - \varepsilon_{c0}} f_c^* & \varepsilon_{c0} \le \varepsilon_c \le \varepsilon_{cu} \\ 0 & \varepsilon_{cu} < \varepsilon_c \end{cases}$$

The strain ε_c and the stress σ_c are positive if they are associated with compression. The initial value of the modulus of elasticity of concrete E_{ci} and the characteristic concrete strength f_c^* ; are assumed known. It is also assumed that the reduction factor $\beta = 0.85$ and that the ultimate concrete strain $\varepsilon_{cu} = 0.0038$. In addition, the following equality holds: $\varepsilon_{c0} = 2\dfrac{f_c^*}{E_{ci}}$.

The constitutive model used to describe the concrete behavior is mono-dimensional, i. e. it does not take into account the existence of a 2-D state of stresses. A more general formulation including these two-dimensional stresses states can be written in the following form:

$$\sigma_{ci} = f_i(\varepsilon_{c1}, \varepsilon_{c2}) \quad i = 1, 2 \tag{3}$$

in which σ_{ci} and ε_{ci} are respectively the principal stresses and the strains of the corresponding tensors acting at the point. Several models have been used with this objective in the technical literature. Most of them can be classified into one of the following groups: (1) Plastic models, (2) Endocronic models and (3) Non linear elastic models.

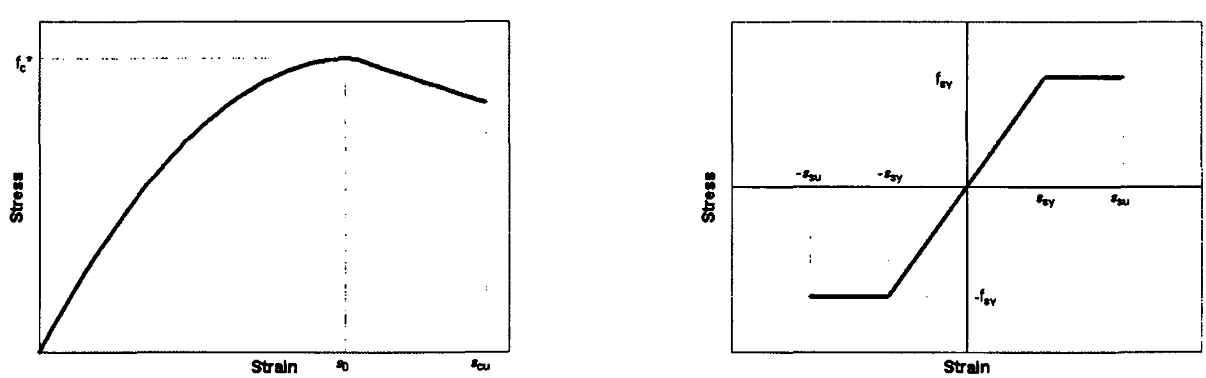

Figure 1: Concrete and steel constitutive behavior

The adjustment of the parameters of these models for the concrete *material* are based on the experimental studies carry out by Kupfer and others [6]. Also it should be mentioned for its special interest the results obtained by Vecchio and Collins [10] using an extensive experimental research in reinforced concrete elements subjected to pure shear stresses. These authors suggest the following relationship that shows the dependence of the maximum tension strain on the principal compression stress of the material *reinforced concrete*:

$$\sigma_{c2} = f_{2max}\left[2\left(\frac{\varepsilon_{c2}}{\varepsilon_{cu}}\right) - \left(\frac{\varepsilon_{c2}}{\varepsilon_{cu}}\right)^2\right] \tag{4}$$

$$\text{with} \quad \frac{f_{2max}}{f_c} = \frac{1}{0.8 + 170\varepsilon_1} \leq 1$$

- Steel

Global equations (5)

$$\sigma_s = \sigma_s(\varepsilon_s) =$$

$$\begin{cases} E_{s1}\varepsilon_s & 0 \leq |\varepsilon_s| < \dfrac{f_{sy}}{E_{s1}} \\ E_{s2}\varepsilon_s \pm f_{sy}\left(1 - \dfrac{E_{s2}}{E_{s1}}\right) & \dfrac{f_{sy}}{E_{s1}} \leq |\varepsilon_s| \leq \varepsilon_{su} \\ 0 & \varepsilon_{su} < |\varepsilon_s| \end{cases}$$

Incremental tangential equations (6)

$$\frac{d\sigma_s}{d\varepsilon_s} = E_{ts} =$$

$$\begin{cases} E_{s1} & 0 \leq |\varepsilon_s| < \dfrac{f_{sy}}{E_{s1}} \\ E_{s2} & \dfrac{f_{sy}}{E_{s1}} \leq |\varepsilon_s| \leq \varepsilon_{su} \\ 0 & \varepsilon_{su} < |\varepsilon_s| \end{cases}$$

The strain ε_s and the stress σ_s are positive if they are associated, respectively, with extension and tension. The modulus of elasticity E_s, the yield stress f_{sy}, the ultimate strain ε_{su} at collapse, and the modulus of elasticity at initial strain hardening E_{s2} are assumed known.

4.4. Design of general bending-membrane reinforcement

We begin by defining some terms. A *shell* is defined as a solid 3-D structure in the form of a surface, with its thickness h small compared to its other dimensions. The middle surface of the shell, in general, possesses double curvature.

At a point of the shell selected for reinforcement evaluation, it will be convenient to introduce a cartesian coordinate system $x_1x_2x_3$. At any point on the middle surface the axes x_1 and x_2 lie in a plane tangent to the middle surface, and the axis x_3 is normal to it (Figure).

As a result of external loads, there exists in the shell a symmetric state of stress σ_{ij} with $i, j = 1, 2, 3$. In the simplest consistent shell theory, Love's first approximation in the Koiter formulation, there are eight independent stress resultants and couples, that are defined as follows ($\alpha\beta = 1, 2$).

In-plane forces or membrane stress resultants

$$N_{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} dx_3$$

Bending stress resultants or normal shear forces

$$Q_\alpha = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{3\alpha} dx_3 \tag{7}$$

Bending stress couples or moments

$$M_{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3\sigma_{\alpha\beta} dx_3$$

The symbols on the left hand sides of equations (7) are used to designate the stress resultants at the point in study obtained from the elastic analysis of the shell. Positive directions of stress resultants and couples are shown in Figure 2

It is of interest to note that, typically, shear stresses associated with stress resultants Q_1 and Q_2 are small, and concrete without reinforcement is capable of transferring them. For this reason, they will not be considered in the analysis presented below. The notation pertaining to reinforcement in the shell is given in the following.

1. I: number of reinforcing bar groups. Each set of bar directions in each curtain of reinforcing is defined as a group
2. i: reinforcing bar group index number
3. A_i (cm^2/cm): area of bars of group i per unit width of shell
4. α_i (radians): angle between bars of group i and the coordinate axis x_1
5. h_i (cm): distance (along coordinate x_3) from the middle surface to the centroid of bar group i

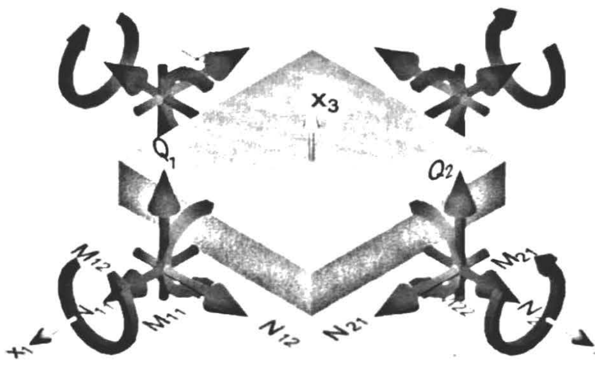

Figure 2: Stresses resultants in a shell

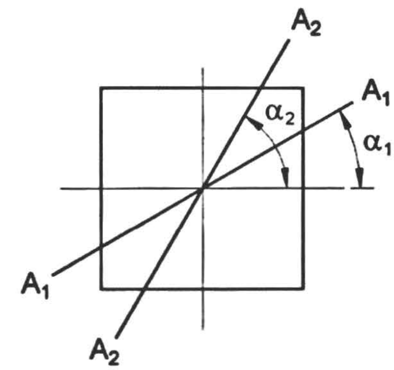

Figure 3: Reinforcement disposition in a plane tangent to a shell

Other symbols were defined in Section 4.3. Additional symbols will be defined when they first occur.

In order to investigate the adequacy of reinforcement at any point of the shell, it is necessary to write down the equations which govern the problem. The constitutive equations have been already given in Section 4.3. The equilibrium equations and the governing equations are discussed in the following.

The state of strain at a point of the shell can be described completely with the aid of just six basic unknowns. We follow Samartín [2] and [1] and choose these to be the following.

- ε_1^0, ε_2^0 with $\varepsilon_1^0 \geq \varepsilon_2^0$, principal in-plane strains of the middle surface at the point under consideration.

- θ^0 (radians) angle between axis Ox_1 and the direction of the normal to the shell cross-section on which the principal strain ε_1^0 is acting. Measured positive anticlockwise from Ox_1.

- κ_1^0 and κ_2^0 with $\kappa_1^0 \geq \kappa_2^0$, principal changes of curvature of the shell element at the point under investigation; each of these curvatures has a positive value if it produces positive work with the corresponding positive principal bending moment, i.e., a curvature causing elongations at the positive region $x_3 > 0$ of the shell.

- φ^0 (radians) angle between axis Ox_1 and the direction of the normal to the shell cross-section on which the principal curvature κ_1 is acting. The angles θ^0 and φ^0 are measured in anticlockwise sense with origin the axis Ox_1 and their variation ranges are $(-\frac{\pi}{2}, \frac{\pi}{2})$.

The six basic unknown variables are collected in a column vector $\mathbf{r}$, i.e.

$$\mathbf{r} = (\varepsilon_1^0, \varepsilon_2^0, \theta^0, \kappa_1^0, \kappa_2^0, \varphi^0)^T$$

The verification of the reinforcement in a concrete shell is allowed for by dividing the shell thickness into a number of layers. A set of governing equations can be written for each layer. Then, the equations of global equilibrium in stress resultants and couples are applied to the whole shell element. The following equations are used in order to obtain the governing equations of the problem.

The concrete strain components $\varepsilon_{\alpha\beta}(x_3)$, with $\alpha, \beta = 1, 2$, existing at the layer of level x_3 are obtained as sum of the strains $\varepsilon_{\alpha\beta}^m(x_3)$ produced by in-plane strains and the strains $\varepsilon_{\alpha\beta}^p(x_3)$ caused by the changes of curvatures, i.e.,

$$\varepsilon_{\alpha\beta}(x_3) = \varepsilon_{\alpha\beta}^m(x_3) + \varepsilon_{\alpha\beta}^p(x_3), \quad \alpha, \beta = 1, 2 \quad (8)$$

The expressions of $\varepsilon_{\alpha\beta}^m(x_3)$ and $\varepsilon_{\alpha\beta}^p(x_3)$ in terms of the basic unknowns are:

$$\begin{aligned} \varepsilon_{11}^m(x_3) &= \varepsilon_1^0 \cos^2\theta^0 + \varepsilon_2^0 \sin^2\theta^0 \\ \varepsilon_{22}^m(x_3) &= \varepsilon_1^0 \sin^2\theta^0 + \varepsilon_2^0 \cos^2\theta^0 \\ \varepsilon_{12}^m(x_3) &= (\varepsilon_1^0 - \varepsilon_2^0)\sin\theta^0 \cos\theta^0 \end{aligned} \quad (9)$$

$$\begin{aligned} \varepsilon_{11}^p(x_3) &= x_3\kappa_1^0 \cos^2\varphi^0 + x_3\kappa_2^0 \sin^2\varphi^0 \\ \varepsilon_{22}^p(x_3) &= x_3\kappa_1^0 \sin^2\varphi^0 + x_3\kappa_2^0 \cos^2\varphi^0 \\ \varepsilon_{12}^p(x_3) &= x_3(\kappa_1^0 - \kappa_2^0)\sin\varphi^0 \cos\varphi^0 \end{aligned} \quad (10)$$

Then, the principal strains $\varepsilon_i(x_3)$, $i = 1, 2$ produced in the layer x_3 and the angle $\theta(x_3$ between the direction of the principal strain $\varepsilon_1(x_3)$ and the axis x_1) are computed according to the standard elasticity formulae:

$$\begin{aligned}
\varepsilon_1(x_3) &= \frac{\varepsilon_{11}(x_3)+\varepsilon_{22}(x_3)}{2}+\Delta \\
\varepsilon_2(x_3) &= \frac{\varepsilon_{11}(x_3)+\varepsilon_{22}(x_3)}{2}-\Delta \\
\tan 2\theta(x_3) &= \frac{2\varepsilon_{12}(x_3)}{\varepsilon_{11}(x_3)-\varepsilon_{22}(x_3)} \\
\text{with} \quad \Delta &= \sqrt{\left[\frac{\varepsilon_{11}(x_3)-\varepsilon_{22}(x_3)}{2}\right]^2+\varepsilon_{12}(x_3)^2}
\end{aligned} \tag{11}$$

Using the constitutive equations of Section 4.3., one can calculate that the principal concrete stress components $\sigma_{c\alpha}(x_3)$ in the layer x_3 are $(\alpha = 1, 2)$:

$$\sigma_{c\alpha}(x_3) = \sigma_c(-\varepsilon_\alpha) \tag{12}$$

Then, the concrete stress components $\sigma_{\alpha\beta}(x_3)$ in the layer x_3 can be found by using standard elasticity transformation formulae:

$$\begin{aligned}
\sigma_{11}(x_3) &= \sigma_c[-\varepsilon_1(x_3)]\cos^2\theta+\sigma_c[-\varepsilon_2(x_3)]\sin^2\theta \\
\sigma_{22}(x_3) &= \sigma_c[-\varepsilon_1(x_3)]\sin^2\theta+\sigma_c[\varepsilon_2(-x_3)]\cos^2\theta \\
\sigma_{12}(x_3) &= \sigma_c[-\varepsilon_1(x_3)]-\sigma_c[-\varepsilon_2(x_3)]\sin\theta\cos\theta
\end{aligned} \tag{13}$$

Therefore, the components of the concrete strength forces $N_{c\alpha\beta}$ and moments $M_{c\alpha\beta}$ are computed by the expressions:

$$N_{c\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{c\alpha\beta}(x_3)dx_3$$

$$M_{c\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3\sigma_{c\alpha\beta}(x_3)dx_3$$

i. e.

Concrete strength forces (14)

$$N_{c11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{\sigma_c[-\varepsilon_1(x_3)]\cos^2\theta+\sigma_c[-\varepsilon_2(x_3)]\sin^2\theta\right\}dx_3$$

$$N_{c22} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{\sigma_c[-\varepsilon_1(x_3)]\sin^2\theta+\sigma_c[\varepsilon_2(-x_3)]\cos^2\theta\right\}dx_3$$

$$N_{c12} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \{\sigma_c[-\varepsilon_1(x_3)]-\sigma_c[-\varepsilon_2(x_3)]\}\sin\theta\cos\theta dx_3$$

Concrete strength moments (15)

$$M_{c11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3\left\{\sigma_c[-\varepsilon_1(x_3)]\cos^2\theta+\sigma_c[-\varepsilon_2(x_3)]\sin^2\theta\right\}dx_3$$

$$M_{c22} = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3\left\{\sigma_c[-\varepsilon_1(x_3)]\sin^2\theta+\sigma_c[\varepsilon_2(-x_3)]\cos^2\theta\right\}dx_3$$

$$M_{c12} = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3\{\sigma_c[-\varepsilon_1(x_3)]-\sigma_c[-\varepsilon_2(x_3)]\}\sin\theta\cos\theta dx_3$$

Similarly, if a perfect bond between concrete and steel bars exists, then the longitudinal strain ε_{si} of the bars group i is:

$$\varepsilon_{si} = \varepsilon_1(h_i)\cos^2[\theta(h_i)-\alpha_i]+\varepsilon_2(h_i)\sin^2[\theta(h_i)-\alpha_i] \tag{16}$$

and then, the axial force S_i of bar i, tension or compression, with $i = 1, 2, \dots I$ is given by the expression:

$$S_i = A_i\sigma_s(\varepsilon_{si}) \tag{17}$$

as evaluated by equation (5).

Therefore, the components of the steel strength forces $N_{s\alpha\beta}$ and moments $M_{s\alpha\beta}$, corresponding to the intensity of axial forces in the bars Si, can be computed using the expressions:

Steel strength forces (18)

$$N_{s11} = \sum_{i=1}^{I} S_i\cos^2\alpha_i$$

$$N_{s22} = \sum_{i=1}^{I} S_i\sin^2\alpha_i$$

$$N_{s12} = \sum_{i=1}^{I} S_i\sin\alpha_i\cos\alpha_i$$

Steel strength moments (19)

$$M_{s11} = \sum_{i=1}^{I} h_iS_i\cos^2\alpha_i$$

$$M_{s22} = \sum_{i=1}^{I} h_iS_i\sin^2\alpha_i$$

$$M_{s12} = \sum_{i=1}^{I} h_iS_i\sin\alpha_i\cos\alpha_i$$

Finally, the force and moment equilibrium equations of the shell element are.

$$\begin{aligned} &N_{s\alpha\beta} - N_{c\alpha\beta} = \lambda N_{\alpha\beta} \\ &M_{s\alpha\beta} - M_{c\alpha\beta} = \lambda M_{\alpha\beta} \\ &\text{with} \quad \alpha, \beta = 1, 2 \end{aligned} \tag{20}$$

in which the negative sign in the concrete strength forces and moments is the result of the sign convention adopted in the constitutive equations of this material. The element ultimate strength at the shell point under consideration is defined by the maximum value of the amplification factor λ for which equilibrium equation (20) holds. This value λ defines also the safety factor related to the collapse load of the shell at the point under consideration.

The six equations of equilibrium (20) can now be expressed in terms of the vector **r** of the basic unknowns by substituting (14), (15),(18) and (19) into (20). The nonlinear resultant system of six simultaneous equations in the unknown variables (vector **r**) has to be solved, in general, by a numerical method, either iterative or incrementally iterative. Details of the procedure can be found in [1].

Once the solution of equations (20) is obtained,, i.e., the basic unknown are found, the values of the remaining variables, namely the strength components of concrete and steel, are computed using equations (14), (15), (18), (19) and (17).

It should be pointed out that, during the evaluation of the integrals over the concrete area, which appear in equations (14) and (15), the angle $\theta = \theta(x_3)$ introduced in expressions (14) and (15) is the angle between the principal strain $\varepsilon_1(x_3)$ at layer x_3 of the shell element and the axis Ox_1. This angle, is not constant, but a continuous function of the coordinate x_3. This coordinate defines the layer across the shell thickness in which the principal strains ε_i, $(i = 1, 2)$ are acting. The computation of the angle $\theta(x_3)$ for each layer x_3 is carried out according to the third equation of the formula (11). In the calculation of the cited integrals discontinuities may appear in the value of the angle θ. This occurs if the strain tensor at the layer x_3 within the thickness $-\frac{h}{2} \leq x_3 \leq \frac{h}{2}$ is critical, i.e., if the following relations hold

$$\varepsilon_{11} = \varepsilon_{22} \quad \text{and} \quad \varepsilon_{12} = 0$$

In this case the continuity at point x_3 can be recovered if the principal stress direction of the tensor are changed, i. e., the value of the angle θ is incremented by $\frac{\pi}{2}$ and the strains ε_i are changed by ε_j with $i, j = 1, 2$ and $i \neq j$.

4.5. Verification of Service and Ultimate Load Limit States

Using the computational procedure described in the previous section, it is possible to find, for each amplification level of the load, an approximation of the crack width, by introducing some simplifying assumptions. To this end, from the known computed positive strain ε_1 normal to the crack direction, and assuming that the crack separation is s, the average crack width a is $a = \varepsilon_1 s$. An estimated value for s is the steel bars separation, that for a shell is typically less than three times its thickness h. Therefore, if the shell thickness h lies between 10 and 25 cm, the expected maximum crack width a, expressed in centimeters, is on the order of 3 to 8 times the strain ε_1.

4.6. Design of membrane reinforcement

A membrane represents a particular case of the general shell. It should be noted that the method for the design of reinforcement described in this section is directly applicable to the design of reinforcement in concrete walls subjected to in-plane loads, a matter of great practical interest to control possible cracking from shrinkage or other causes.

Only the in-plane stress resultants existing in this case, namely $\mathbf{N} = (N_{ij})$ with $i, j = 1, 2$. At any point of the shell, the values of these stress resultants are assumed known. All other stress resultants are assumed to vanish identically. Then, the shell can be represented by a single layer and a significant simplification of the problem can be obtained.

The principal in-plane stress resultants are denoted by N_1 and N_2. There are three possible states of stress, depending on the sign of the principal stress resultants.

1. N_1 and N_2 are both compression
2. One of the principal stress resultants,N_1 or N_2 is tension
3. Both N_1 and N_2 are tensions. As a rule, the geometry of a concrete shell roof should be such that this case does not arise.

The membrane reinforcement is the reinforcing associated with the in-plane state of stress of the membrane shell structure. In the last two cases, verification of the reinforcement is required. In the first case, in principle, no reinforcement is needed. Nevertheless, it should be provided as a matter of good practice.

The equations which govern the design of reinforcement in membrane shells, can be obtained directly from the equations of a general shell.

First, at any point of the middle plane of the shell the stresses are uniformly distributed across the shell thickness and all shell layers behave identically. Then, no bending strains exist and therefore the three basic unknowns associated to these strains are identically zero and the moments equilibrium equations are void.

Therefore only three basic unknowns are adopted, namely, the principal longitudinal strains in the concrete ε_1 and ε_2 and the angle θ between the principal direction of ε_1 and the axis Ox_1. Sign convention follows the same rules as in previous Section 4.4. In case that ε_1 is extension, i. e. $\varepsilon_1 \geq 0$), the angle θ coincides with the one between the fissures direction and the axis Ox_2 (Figure 4).

The force equilibrium equations in a differential membrane element $dx_1 dx_2$, at a specific point, correspond to the first group of the equations (20), i.e., they can be written as follows.

$$\lambda N_{ij} = N_{cij} + N_{sij} \quad \text{with} \quad i, j = 1, 2 \qquad (21)$$

where the stress resultant components related to concrete and steel are designated N_{cij} and N_{sij} respectively, and they are represented in Figures 4 and 5. The expressions for these forces are given by equations (14) and (18).

Equations (21) are a system of simultaneous nonlinear equations in the basic unknowns $\varepsilon_1, \varepsilon_2$ and θ. These equations can be explicitly expressed on these unknowns as follows.

$$\begin{aligned}
& - h\sigma_c(-\varepsilon_1)\cos^2\theta - h\sigma_c(-\varepsilon_2)\sin^2\theta \\
& + \sum_{i=1}^{I} S_i(\varepsilon_1, \varepsilon_2, \theta)\cos^2\alpha_i = \lambda N_{11} \\
& - h\sigma_c(-\varepsilon_1)\sin^2\theta - h\sigma_c(-\varepsilon_2)\cos^2\theta \\
& + \sum_{i=1}^{I} S_i(\varepsilon_1, \varepsilon_2, \theta)\sin^2\alpha_i = \lambda N_{22} \qquad (22) \\
& - [h\sigma_c(-\varepsilon_1) - h\sigma_c(-\varepsilon_2)]\sin\theta\cos\theta \\
& + \sum_{i=1}^{I} S_i(\varepsilon_1, \varepsilon_2, \theta)\sin\alpha_i\cos\alpha_i = \lambda N_{12}
\end{aligned}$$

in which the steel axial force of the bars of the group i is

$$S_i(\varepsilon_1, \varepsilon_2, \theta) = A_i\sigma_s[\varepsilon_1\cos^2(\theta - \alpha_i) + \varepsilon_2\sin^2(\theta - \alpha_i)]$$

in which $i = 1, 2, \ldots I$ andσ_s is evaluated with equation (5).

The system of equations (22) is typically solved numerically, either by iterative procedures [7] or by incremental iterative procedures [1]. The equations are essentially the same as those obtained by Medwadowski [7] and [8].

Reinforcement in the form of a two-way orthogonal mesh constitutes a special case, which is important in practical applications. In this case, the governing equations can be simplified, and the solution of the equilibrium equations (22) can be achieved by introducing additional assumptions regarding the constitutive equations. These simplifications are.

- Concrete is linearly elastic with Young modulus E_c
- Steel is linearly elastic-perfectly plastic material ($E_{s2} = 0$ and $E_{s1} = E_s$)
- One of the principal stress resultants is tension ($\varepsilon_1 > 0$)

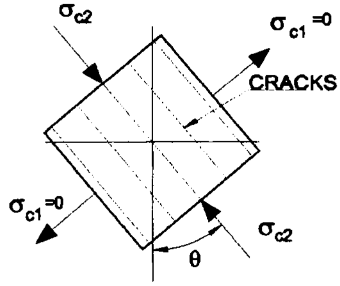

Figure 4: Components of the concrete resistant forces

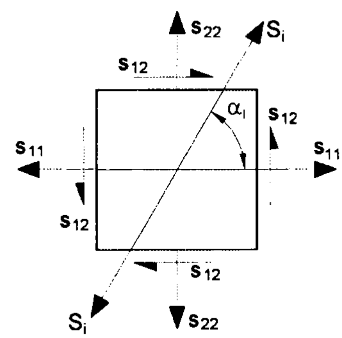

Figure 5: Components of a steel bar resistant forces

The directions of the coordinates axis coincide with the directions of the two reinforcement groups, i. e. $\alpha_1 = 0$ and $\alpha_1 = \frac{\pi}{2}$. Then, the solution of (22) is found as follows: *Case 1*: First, assume that the steel bars are in the elastic range

- Solve the following equation for the unknown $t = \tan\theta$

$$n_{12}a_2(a_1 - 1)t^4 - n_{11}a_2t^3 + n_{22}a_1t - n_{12}a_1(a_2 - 1) = 0 \quad (23)$$

in which

$$a_1 = \frac{A_1E_s}{hE_c}, \quad a_2 = \frac{A_2E_s}{hE_c}, \quad n_{ij} = \frac{N_{ij}}{hE_c}$$
with $i, j = 1, 2$

- Once the solution of equation (23) is found, the two remaining basic unknown values are

$$\varepsilon_1 = \lambda A\frac{1+t^2}{t}, \quad \varepsilon_2 = \lambda n_{12}\frac{1+t^2}{t}A$$
with
$$A = \frac{t(n_{11}+n_{22}) - n_{12}[(a_1-1)t^2 + (a_2-1)]}{a_1 + a_2t^2}$$

if $\varepsilon_2 < 0$ then the shell is subjected at the point under consideration to a bi-tensional state of stress, and this case should be avoided. Also if $-\varepsilon_2 > \varepsilon_{c0}$, the shell thickness should be increased.

- The stress resultants corresponding to the concrete strength are

$$N_{c11} = hE_c\varepsilon_2\sin^2\theta$$
$$N_{c22} = hE_c\varepsilon_2\cos^2\theta$$
$$N_{c12} = -hE_c\varepsilon_2\sin\theta\cos\theta$$

- The corresponding steel strength stress resultants are

$$N_{s11} = E_sA_1(\varepsilon_1\cos^2\theta + \varepsilon_2\sin^2\theta)$$
$$N_{s22} = E_sA_2(\varepsilon_1\sin^2\theta + \varepsilon_2\cos^2\theta)$$
$$N_{s12} = 0$$

The steel bar forces must remain in the elastic range:

$$|N_{s11}| \le f_{sy}A_1, \quad |N_{s22}| \le f_{sy}A_2$$

Case 2: One of the two steel bars, say bar i, has reached yield. In this case equation (23) must be modified by changing a_i for $\bar{a}_i = \frac{A_if_{sy}}{hE_c}\frac{N^0_{sii}}{|N^0_{sii}|}$ with N^0_{sii} the ultimate axial force of the reinforcement group i computed under the previous hypothesis, i. e. as it was in the elastic range. The corresponding ultimate axial force of bar i is now $N_{sii} = f_{sy}A_i\frac{N^0_{sii}}{|N^0_{sii}|}$.

Case 3: The two reinforcement groups of steel bars have reached yield. The equation (23) should be modified by introducing $\bar{a}_i = \frac{A_if_{sy}}{hE_c}\frac{N^0_{sii}}{|N^0_{sii}|}$ instead of a_i for $i = 1, 2$. The axial forces in the steel bars are $N_{sii} = f_{sy}A_i\frac{N^0_{sii}}{|N^0_{sii}|}$ and $N_{s12} = 0$.

4.7. Design of pure bending reinforcement

We consider next the case when membrane forces are assumed to vanish $N_{ij} = 0$ for $i, j = 1, 2$. and only the flexural effects are present - the shell is said to be in the state of *pure bending*. This case is very important in reinforced concrete plate design. The reinforcement has to be designed to resist the bending and torsional moments in the plate.

The governing equations of the case of pure bending can be obtained directly from the equations of the general case discussed in Sections 4.4. and 4.5., with the constitutive equations being the same as those of Section 4.3.

The equilibrium equations can be simplified as follows.

Force equilibrium

$$\lambda N_{ij} = 0 = -N_{cij} + N_{sij} \quad (24)$$

Moment equilibrium

$$\lambda M_{ij} = -M_{cij} + M_{sij} \quad (25)$$

and $i, j = 1, 2$. As in the general case, the number of basic unknowns is six, the three associated with the in-plane strains $\varepsilon_1, \varepsilon_2$ and θ and the three associated with the changes of curvatures, κ_1, κ_2 and φ.

The system of equations (24) and (25) can be expressed explicitly in terms of these basic unknowns. The result is a system of six nonlinear equations in the six basic unknowns. However, in the case of pure bending, it is possible to reduce the number of equations from six to just three. This can be achieved by expressing the in-plane strains in terms of the three curvatures using equations (24) and then substituting the resulting expressions into the three remaining equations (25).

As before, a numerical incremental-iterative solution procedure can be employed. The process of verification of adequacy at service and ultimate limit states follow the same pattern as in this case of the membrane. The only difference is that now the strain to be considered at the service limit state correspond to the maximum strain, i.e., to the greater strain of the two principal strains $\varepsilon_1(\frac{h}{2})$ and $\varepsilon_1(-\frac{h}{2})$ occurring on the two faces of the plate.

5. CONCLUDING REMARKS

Presented in this paper is an overview of the problem of design/verification of reinforcement in concrete shells. The types of reinforcement design are examined and classified. Particular attention is paid to the *elastic-plastic* approach, in which the structure under working loads is analyzed on the assumption that it is elastic, and then, the reinforcement is designed assuming inelasticity of the materials, at appropriate load levels. This approach is widely used in the design of all types of concrete structures.

The procedures for elastic-plastic design/verification of reinforcement in membrane shells have been available for some time [7], [8]. Very recently, they were extended to the case of general shell subjected to both bending and membrane stresses [2] and [1].

These design procedures are described in the remainder of the paper. The general procedure [2] and [1] is discussed first, applicable to the case of a shell subjected to both bending and membrane effects.

This is then reduced to the case of a membrane shell. Finally, the case of pure bending is considered, again by reducing appropriately the general equations.

While the motivation for this overview was primarily the design of reinforcement in concrete shell roofs, the procedures we described apply to other types of concrete shells, and to other structures. We note that the case of membrane shell is the same as the case of a concrete wall subjected to in-plane state of stress, while the case of pure bending is directly applicable to the design/verification of reinforcement in concrete plates subjected to flexure.

In applications, except possibly for the simplest case of a membrane shell with two-way orthogonal reinforcement, the procedure requires the use of a desk-top computer. However, with current technology this is not a significant obstacle to its use.

REFERENCES

[1] Díaz-Dominguez, P., García-Palacios, J., and Samartín, A. An Unified Approach to Verify the Reinforcement in Two-Dimensional Reinforced Concrete Structures at Service and Ultimate Loading. In *Proceedings of the Ninth International Conference on Civil and Structural Engineering Computing*. Egmond-aan-Zee. The Netherlands, Sept. 2-4. 2003.

[2] García-Palacios, J. and Samartín, A. Un método unificado de verificación de las armaduras en estructuras bidimensionales de hormigón armado. *Informes de la Construcción*, 2002.

[3] Gupta, A. K. Membrane reinforcement in shells. *Journal of Structural Division, ASCE*, 107(ST1):41–56, 1981.

[4] Gupta, A. K. and Akbar, H. Cracking in reinforced concrete analysis. *Journal of Structural Engineering, ASCE*, 110(8):1735–1746, 1984.

[5] Hognestad, E. A Study of Combined Bending and Axial Load in Reinforced Concrete Members. Report 1, University of Illinois, Engineering Experiment Station, Bulletin Series No 399, November 1951.

[6] Kupfer, H. B. and Gerstle, K. H. Behavior of concrete under biaxial stresses. *Journal of Engineeering Mechanics Division, ASCE*, 99:EM4, August 1973.

[7] Medwadowski, S. J. Multidirectional membrane reinforcement. *ACI Structural Journal*, 86(5):563–569, 1989.

[8] Medwadowski, S. J. Variation in the strength of membranes with the direction of the reinforcement. *Bulletin of the International Association for Shell and Spatial Structures*, 31(1-2):93–97, April-August 1990.

[9] Nielsen, M. P. On the strength of reinforced concrete discs. Report 70, ACTA Polytechnica Scandinavia, Civil Engineering and Building Construction Series. Copenhagen, 1971.

[10] Vecchio, F. J. and Collins, M. P. The modified compression field theory for reinforced concrete elements subjected to shear. *ACI Journal*, 83(2):219–231, Mar.-Apr. 1986.